Asymptotic Analysis of Die Flow for Shear-Thinning Fluids

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The flow in a single-cavity extrusion die is considered by using 1-D equations. Since the fluid flow in a well-performing die does not deviate much from the case of perfect widthwise distribution, linearization of 1-D governing equations is often justified. It is demonstrated how linearization can lead to significant simplifications in the analysis of die flow. A self-consistent asymptotic expansion is identified to accomplish this linearization; the expansion can be generalized to obtain higher-order corrections if necessary. In contrast to previous analyses, which have generally been numerical, analytic solutions are obtained to predict widthwise flow nonuniformities in a die having a specific cavity area variation and in which cavity inertial effects are important; these solutions are valid for power-law and Newtonian fluids. A novel truncated power-law approach to modeling the flow of shear-thinning fluids in a die is proposed that combines, by using analytical criteria, the solutions for purely Newtonian or power-law flow in the cavity and slot. The validity of this approach is demonstrated through comparison with numerical results of a 1-D die model for a Carreau constitutive equation.

Introduction

The role of an extrusion die is to create a 2-D liquid film that is wider than its thickness. These films are often formed as an intermediate step in the manufacture of polymeric sheets and photographic products and may require a high degree of thickness uniformity. Figure 1 shows a typical extrusion die consisting of a cavity and a slot, both of width W. The cavity itself is a distribution chamber, redistributing fluid from an inlet to the ultimate width W; fluid is extruded from the slot to form the liquid film. To achieve high uniformity in the widthwise direction (the z-direction in Figure 1a), the resistance to flow in the cavity is made low by choosing a relatively large cross-sectional area, A (Figure 1b), while the slot is designed such that its resistance to flow is high; this is accomplished by choosing a small slot height, h, and a relatively long slot length, L. In this way, fluid entering the distribution chamber tends to distribute widthwise before entering the slot; consequently, the flow in the slot is oriented primarily toward the slot exit (upward in Figure 1).

In an extrusion die having a cavity and slot with dimensions that are independent of the z-direction in Figure 1a, perfect widthwise uniformity cannot be accomplished, owing in part to the fact that a constant cavity pressure cannot be obtained with a finite-sized cavity. There must be a pressure drop over the cavity width W to force the fluid to distribute widthwise. Various geometric adjustments can be utilized to counteract these pressure variations, such as widthwise varying cavity areas (Figure 1a), slot-length variations, and even an additional die cavity (such dies are called dual-cavity designs); however, such procedures can only be used to assure flow uniformity for one fluid type and flow rate. In many cases, however, a single extrusion die is called on to deliver a variety of liquids having a high degree of widthwise uniformity. The goal of a die designer is to be able to deliver, with a single die, the largest range of fluid flows and fluid types to within specified uniformity limits. To this end, a model for flow in dies is a valuable tool.

A typical coating die is wide and corresponds to cases where the slot length is smaller than its width, as $L \ll W$ in Figure 1a. As discussed by Weinstein and Ruschak (1996), the principal theoretical analyses used in die design for such cases are 3-D computation or 1-D modeling. (Note that a 2-D modeling approach can be used to model flows in "narrow" dies for which the $L \ll W$ assumption is not satisfied; see, for example, Wen and Liu, 1995.) When 3-D computation is used, the complete mass and momentum conservation equations are solved to obtain a pointwise accounting of the fluid velocities and pressures in the die. Alternatively, in the 1-D modeling approach, the 3-D equations (those used in 3-D computation)

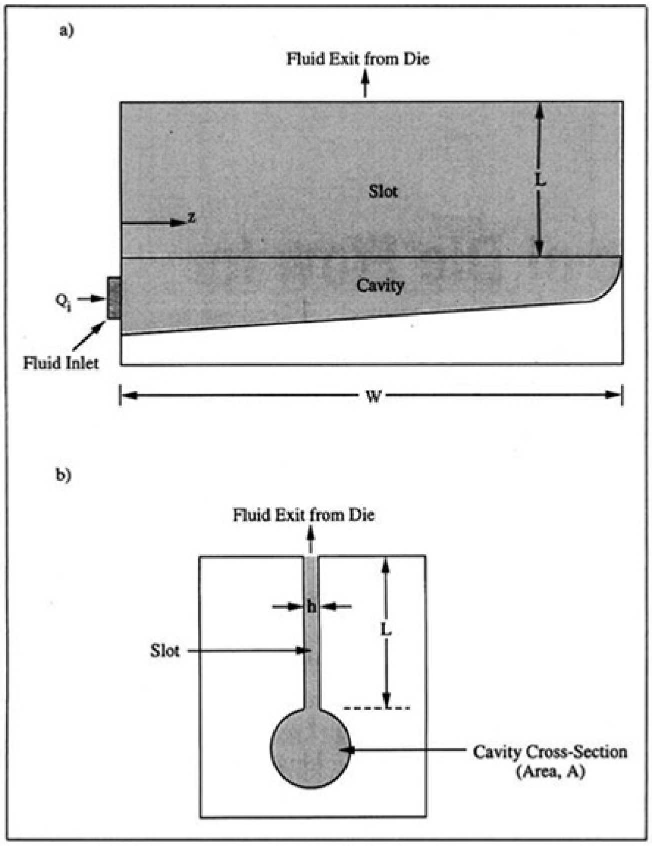

Figure 1. Typical geometry of a single-cavity extrusion die.

(a) Widthwise view; (b) side view.

are approximated with a simpler set of equations, where all variables depend only on the widthwise location z in Figure 1. The advantage of such simplified equations is that they are much easier to solve, and many flow conditions and die geometries can be investigated relatively quickly. Of course, since the 1-D equations are approximate, there may be some adjustment necessary to hopper designs obtained. These adjustments can be made by utilizing more precise and intensive 3-D computation. The combined use of 1-D modeling and 3-D computation is advantageous in that it can reduce the time to design a die. In this article, attention is focused on 1-D modeling.

One-dimensional modeling of dies has been used extensively in previous studies, and we make no attempt to provide a complete review here. Much of the previous work has focused on single-cavity die flows having both negligible fluid inertia and a power-law viscosity dependence (Carley, 1954; Pearson, 1964; McKelvey and Ito, 1971; Chung and Lohkamp, 1976; Winter and Fritz, 1986; Liu et al., 1988; Gutoff, 1991). Leonard (1985) has extended these analyses to include the effects of fluid inertia in the cavity (gravitational effects are included as well). As discussed extensively by Weinstein and Ruschak (1996), various forms of the cavity momentum equation have been proposed for single-cavity dies when inertial effects are incorporated [compare, for example, the cavity equations of Leonard (1985), Sartor (1990), and Durst et al. (1994)]. Additionally, when dual-cavity dies have been examined, the inner-cavity (the cavity connected to the inlet) equation should be identical to that for the single-cavity die, but again different cavity equations have been proposed (Lee and Liu, 1989; Yuan, 1995). By formalizing the derivation of the cavity equation, Weinstein and Ruschak (1996) have rectified the differences between the various proposed cavity analyses; additionally, they have placed all the single-cavity, 1-D equations on firmer theoretical ground, identifying the specific assumptions implicit in their use. Weinstein and Ruschak show that the equations proposed by Leonard (1985) are valid for typical single-cavity extrusion dies. We comment here that the constant parameters in the model of Leonard (1985), the so-called shape factors that specifically depend on the cavity shape and power-law index, have been generated by many (Miller, 1972; Liu, 1983; Lee and Liu, 1989).

In utilizing the 1-D equations to obtain single-cavity designs for power-law fluids, there have been two general approaches utilized. In the first approach, the local variation in the widthwise (in the z-direction in Figure 1) volumetric flow extruded from the slot is predicted for a given die geometry. Because of typical die geometry variations (such as shown in Figure 1a), coupled with fluid inertia effects that introduce nonlinear terms in the cavity momentum equation, numerical predictions of slot-flow uniformity have typically been employed (Leonard, 1985; Liu et al., 1988; Wu et al., 1994; Durst et al., 1994) except for analyses having simplified geometries and no inertia (Carley, 1954). In the second widely used approach, geometric adjustments necessary to achieve "perfect" widthwise uniformity in the slot are determined. For analyses in which inertial effects are neglected, the second approach has generally afforded the prediction of slot and cavity dimensions via analytical means (Pearson, 1964; Chung and Lohkamp, 1976; Winter and Fritz, 1986; Gutoff, 1991). Once inertial effects are included in such analyses, the geometric adjustments have been made numerically. A drawback of the second approach is that once a die geometry has been identified to achieve perfect uniformity for a given flow condition and fluid rheology, it is not possible to predict the die performance for different fluids through the die (this must be done with the first approach); rather, in the second approach, one is relegated to showing that the geometrical die design to achieve perfect uniformity does not vary much for a range of fluid types and flows (for example, Winter and Fritz, 1986). Nevertheless, the second approach can clearly be utilized to help limit the parameter space for a reasonable design.

An inherent feature of just-cited literature is that the desired die design is one in which deviation from perfect uniformity is small (or zero), and in most cases, for a range of fluids. Despite this fact, there are few studies in which solutions of the 1-D equations have been obtained as small deviations from the perfect uniformity case: they have used a linearization approach to predict the performance of dies. Carley (1954) used linearization to obtain an analytic solution for flow nonuniformities arising in a single-cavity die in which the cavity had a constant circular cross-sectional area and constant slot dimensions; the fluid was assumed to have a power-law dependence, and inertia was neglected in the cavity. More recently, Durst et al. (1994) examined die performance for power-law fluids and where fluid inertia was not neglected; geometries having constant cavity shapes and areas as well as slot-length variations were examined. Durst et al. used linearization ideas in a "qualitative analysis" of the slot-length variation and obtained an estimate for design situations in which the slot length did not need to vary to achieve highly uniform slot flows. Additionally, in obtaining a numerical solution of their governing equations for slot-flow nonuniformity, Durst et al. (1994) utilized a linearized form of the governing equations (linearized about the perfect uniformity case) to achieve a simpler and less intensive numerical algorithm. Durst et al. demonstrated the validity of the linearization approach by showing agreement with the nonlinear analysis of Sartor (1990) for various cases.

One common aspect of the previously cited analyses of die flows is that a power-law dependence for the viscosity has been assumed (except, as discussed below, for the work of Yuan, 1995). We now discuss this assumed viscosity form in more detail. For power-law fluids, the viscosity, η, is related to the magnitude of the rate of strain $|\dot{\gamma}|$, as

$$\eta = m|\dot{\gamma}|^{n-1}, \tag{1}$$

where m is the consistency coefficient and n is the power-law index (Bird et al., 1977). Figure 2a shows a comparison of a typical experimental viscosity-strain dependence (the rheological curve) and the power-law viscosity dependence. For the rheology shown in Figure 2a, the experimental viscosity tends to decrease as the rate of strain increases, that is, the fluid exhibits "shear-thinning" rheology. It is clear that, while the rheological curve approaches constant Newtonian viscosities (i.e., they are strain-independent) at low and high rates of strain, the power-law model is inadequate to capture such behavior, although the model does capture the region of appreciable shear thinning quite well; for this reason, the shear-thinning region of Figure 2a is often referred to as the power-law region. As pointed out by Yuan (1995), the shortcomings of the power-law model can make it inadequate in predicting flow nonuniformities for fluids that we may classify as being moderately shear thinning. For such fluids, the flow may exhibit "Newtonian" behavior in portions of the cavity but, at the same time, the flow in the slot may exhibit "power-law" rheology; such a situation can arise due to the disparate rates of strain in the cavity and slot of a die. Obviously, for cases in which the fluid shear thins even at the disparate strains in the cavity and slots (which we can term as highly shear-thinning fluids), the power-law viscosity dependence is likely a reasonable model.

There are many rheological models that do overcome the shortcomings of the power-law model; one such representative model is that of Carreau (Bird et al., 1977), whose viscosity dependence is given by

$$\eta - \eta_\infty = (\eta_0 - \eta_\infty)\left[1 + (\lambda|\dot{\gamma}|)^2\right]^{(n-1)/2}, \tag{2}$$

where η_0 and ${}_{h\infty}$ are the respective Newtonian limiting viscosities at low and high rates of strain (Figure 2a), λ is the relaxation time, and n controls the slope of the power-law region of the rheological curve. The Carreau model provides an excellent fit to the experimental curve in Figure 2a. As demonstrated by Weinstein and Ruschak (1996), however, the Carreau model cannot be easily implemented in the context of the one-dimensional approach; this is because constant shape factors cannot be obtained for given cavity shapes and

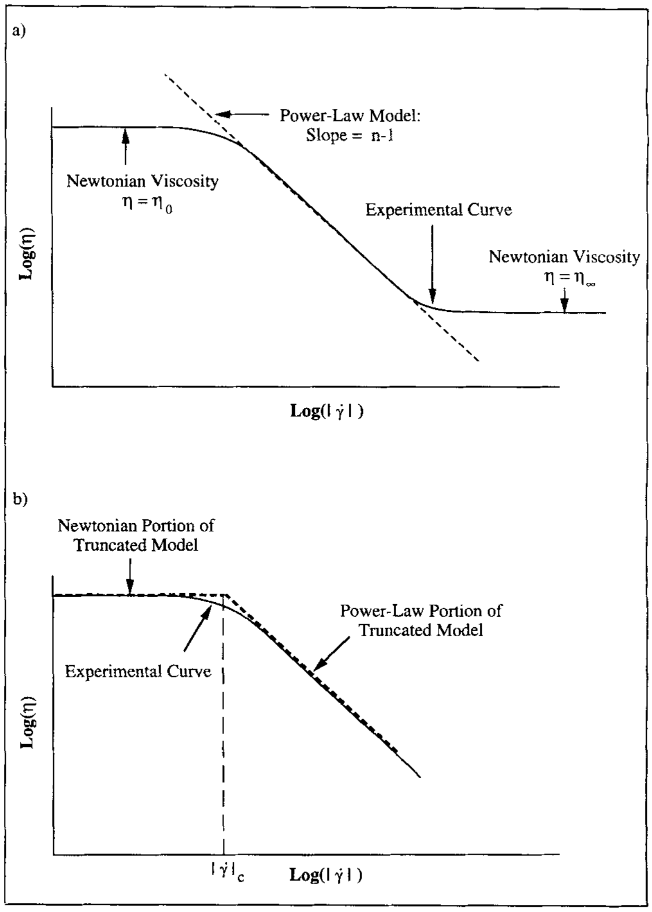

Figure 2. Dependence of the viscosity, η, on the magnitude of the rate of strain, $|\dot{\gamma}|$, for a shear-thinning fluid.

(a) Typical experimental rheology curve; (b) relationship between the truncated power-law model and the actual rheology curve for $\eta > {}_{h\infty}$.

rheologies. Constant shape factors *can* be obtained for power-law fluids and Newtonian fluids, and simplified one-dimensional equations can thus be defined for both fluid types. For this reason, Yuan (1995) utilizes a truncated power-law model (Bird et al., 1977) in his analysis, which combines Newtonian and power-law asymptotes to the experimental curve in Figure 2a. For situations in which $\eta \gg {}_{h\infty}$ (which may be satisfied everywhere in a die due to the high rates of strain that may be required to achieve $\eta = {}_{h\infty}$), the truncated model is given by

$$\eta = \eta_0 \quad \text{for} \quad |\dot{\gamma}| \le |\dot{\gamma}|_c \tag{3a}$$

$$\eta = m|\dot{\gamma}|^{n-1} \quad \text{for} \quad |\dot{\gamma}| \ge |\dot{\gamma}|_c, \tag{3b}$$

where $|\dot{\gamma}|_c$ is the critical rate of strain at which the Newtonian and power-law behaviors are linked; that is, $\eta = \eta_0$ in Eq. 3b, and thus

$$|\dot{\gamma}|_c = \left(\frac{\eta_0}{m}\right)^{1/(n-1)} \tag{3c}$$

A plot of the truncated model (Eq. 3) is shown in Figure 2b.

In the numerical analysis of Yuan (1995), the average rate of strain at each widthwise location (z in Figure 1) in the slot and cavity regions is examined, and a choice of power-law or Newtonian rheology (and the associated one-dimensional equation) is made for these locations according to Eq. 3.

In this article, we demonstrate how simplifications made to the one-dimensional equations through linearization can be used to address the rheology issues previously discussed. The linearization of the one-dimensional equations can be accomplished formally via an asymptotic expansion, which can be generalized to obtain higher order corrections if necessary. In contrast to previous work, which has generally been numerical, the asymptotic approach can often yield analytical solutions for the flow nonuniformities that arise; in this article, one such solution is obtained for a particular cavity area variation and constant slot dimensions. Furthermore, we utilize a truncated power-law approach to handle moderately shear-thinning fluid rheology; however, the implementation of our approach is different than the numerical approach of Yuan (1995). In particular, in our approach, we demonstrate how analytic criteria, called "switching criteria," can be derived that allow for the determination of whether the flow can be effectively viewed as being Newtonian or power law in the cavity of the die *as a whole*; the same is done for the slot flow. There is no local z-dependent alternation between the power-law and Newtonian models (using Eq. 3), as in the analysis of Yuan (1995). Since it is possible to choose the appropriate cavity and slot-flow models (power-law or Newtonian) *a priori*, our approach is straightforward to utilize.

This article is organized as follows. First, we consider the asymptotic approximations to the 1-D equations for power-law and Newtonian flows in a single-cavity die. We then obtain a general analytical expression for slot-flow nonuniformities for a particular cavity area variation. Next, the asymptotic result is used to obtain switching criteria (referred to earlier) that allow us to combine Newtonian and power-law analyses to predict flow nonuniformities in an extrusion die. Finally, we assess the validity of the asymptotic approach and the switching criteria by comparing predicted nonuniformity results with numerical results of a one-dimensional die model for a Carreau fluid; this comparison is made for a die geometry consisting of a straight slot and circular cavity, to simplify the Carreau numerical analysis. The Carreau analysis itself, being numerically intensive even for the simple geometry examined, also serves to demonstrate why it is prohibitive to use it in the context of a one-dimensional approach.

Asymptotic Approximation and Flow Nonuniformity Solution

The 1-D equations governing flow in a die are given by Leonard (1985); in this article, we use the notation of Weinstein and Ruschak (1996), who derived an identical set of equations to those of Leonard. In the context of the 1-D approach, the problem specification is as follows. We consider the flow of a fluid having density ρ in a single-cavity die of width W, whose geometry is shown in Figure 1. The inlet and end of the die are located at $z=0$ and $z=W$, respectively. The cavity of the die has a constant cross-sectional shape, although the cross-sectional area, A, can vary widthwise in the z-direction. The length of the slot, L, and the slot height, h, are assumed constant. The z-dependent volumetric flow and pressure in the cavity are denoted by Q and P, respectively; the z-dependent volumetric flow per width in the slot is q. Let us denote the inlet volumetric flow rate, introduced at $z=0$, as Q_i, and denote the cross-sectional area at $z=0$ as A_i. We neglect all gravitational effects in our analysis.

The fluid viscosity is assumed to obey a power-law model as given by Eq. 1. As stated in the introduction, however, we anticipate utilizing a combined power-law and Newtonian approach to model the fluid flow in the die; in this approach, a decision is made for the cavity (as a whole) and the slot (as a whole) as to whether the flow is Newtonian or power law in these regions. We will derive criteria later to make this decision; however, we make two adjustments to the power-law model here to facilitate the combined approach. First, we relate the parameters of the Carreau and power-law models as follows. For cases where $\eta \gg {}_{h\infty}$ and $(\lambda|\dot{\gamma}|)^2 \gg 1$, Eq. 2 becomes $\eta \sim (\eta_0 \lambda^{n-1})|\dot{\gamma}|^{n-1}$. By inspection of Eq. 1, the relationship between the Carreau and power-law parameters is thus given by

$$m = \eta_0 \lambda^{n-1} \qquad \text{when} \quad \eta \gg {}_{h\infty}. \tag{4a}$$

In the truncated model (Eq. 3), this relationship implies that the critical rate of strain for the onset of shear thinning is

$$|\dot{\gamma}|_c = \frac{1}{\lambda}. \tag{4b}$$

Consequently, the parameter λ in the Carreau model can be viewed as being the approximate rate of strain above which appreciable shear thinning occurs; it is thus a natural parameter to utilize in determining whether a shear-thinning fluid flows as power law or Newtonian in a particular region of the die. Note that in using Eq. 4a, we implicitly assume in this article that the viscosity satisfies $\eta \gg {}_{h\infty}$ everywhere in the die. The second adjustment to the power-law model is made by introducing notation to distinguish between the power-law indices in the equation governing the cavity flow and that governing the slot flow. Let us denote n_s and n_c as the effective power-law indices in the cavity and slot, respectively; these parameters can separately take on the values of 1 or n, depending on whether the flow in the slot or cavity is Newtonian or power law. By using this notation, we can solve the one-dimensional equations once and then use a single mathematical result for various combinations of power-law and Newtonian flows in the cavity and slot of the die.

With the above-described problem specification and modifications to the power-law model, we scale the equations of Weinstein and Ruschak (1996) as follows. The cavity cross-sectional area scales as $A \sim A_i$. The widthwise variable z scales as $z \sim W$, and the slot flow per width scales as $q \sim Q_i/W$, that is, the ideal flow per width. Let us also choose the pressure scale based on the Newtonian pressure drop across the slot at $z=0$, that is, $P \sim 12\eta_0 L Q_i/(Wh^3)$, where η_0 is the Newtonian viscosity. Defining all dimensionless variables with an overbar, the one-dimensional equations are:

Cavity Momentum Equation:

$$\frac{d\bar{P}}{d\bar{z}} = -\epsilon\left[\frac{\bar{Q}^{n_c}}{\bar{A}^{(1+3n_c)/2}} + Re\,\frac{1}{\bar{A}}\frac{d}{d\bar{z}}\left(\frac{\bar{Q}^2}{\bar{A}}\right)\right]. \tag{5a}$$

Slot Momentum Equation:

$$\bar{P} = \alpha \bar{q}^{n_s}. \tag{5b}$$

Mass Conservation:

$$\frac{d\bar{Q}}{d\bar{z}} = -\bar{q}. \tag{5c}$$

Boundary Conditions:

$$@\ \bar{z}=0, \quad \bar{Q}=1; \quad @\ \bar{z}=1, \quad \bar{Q}=0, \tag{5d}$$

where

$$\epsilon = \left(\frac{W^2h^3}{12LA_i^2K^{n_c}}\right)\left(\frac{\lambda Q_i}{A_i^{3/2}}\right)^{n_c-1},$$

$$Re = \left(\frac{\rho\beta Q_i K^{n_c}}{\eta_0 W}\right)\left(\frac{\lambda Q_i}{A_i^{3/2}}\right)^{1-n_c},$$

$$\alpha = \frac{2^{n_s}}{6}\left(\frac{1}{n_s}+2\right)^{n_s}\left(\frac{\lambda Q_i}{Wh^2}\right)^{n_s-1}. \tag{5e}$$

In Eq. 5e, K and β are the viscous and kinetic cavity shape factors, respectively (see Weinstein and Ruschak (1996) for their precise definitions). These shape factors quantify the dimensionless cavity flow rate and inertia for a given cavity shape (even if the area of the cavity varies) and power-law index n_c; typical values have been generated by many (Miller, 1972; Liu, 1983; Lee and Liu, 1989).

The goal is to solve for $\bar{Q}$, $\bar{q}$, and $\bar{P}$, which are all functions of z. Of particular importance is the determination of the slot-flow rate per width, $\bar{q}$; variations in this quantity determine the effectiveness of the die in delivering a uniform flow. In dimensionless terms, the uniformity index, N, which represents the fractional variation in the slot-flow quantity q, is given by

$$N = \bar{q} - 1. \tag{6}$$

Note that when there is perfect flow uniformity, $\bar{q}=1$, and thus $N=0$ at each point widthwise across the die. In practice, designs leading to perfect uniformity are difficult to attain, especially so for a wide range of fluids and flow rates. The practical goal in die design, then, is to minimize the deviation of $\bar{q}$ from 1 in Eq. 6, over some broad range of conditions.

We now consider the requirements necessary for an acceptable design, as quantified by the parameters found in Eq. 5e. By inspection of Eq. 5b, it is seen that an ideal design arises when the pressure in the cavity is an absolute constant, since this leads to a constant value of $\bar{q}$ (which must be equal to 1 from a mass balance). Noting that the parameter α is on the order of unity (e.g., for $n=1$, $\alpha=1$), it is clear from Eq. 5a, that designs having $\epsilon \ll 1$, and $\epsilon Re \ll 1$, will yield flows approaching the ideal, constant cavity pressure case. These small-parameter values suggest that the system (Eq. 5) can be simplified using asymptotic techniques by considering the limit as $\epsilon \to 0$, holding Re and α fixed. To this end, we expand the dependent variables as

$$\bar{Q} \sim \bar{Q}_0 + \epsilon\bar{Q}_1 + O(\epsilon^2) \tag{7a}$$

$$\bar{P} \sim \bar{P}_0 + \epsilon\bar{P}_1 + O(\epsilon^2) \tag{7b}$$

$$\bar{q} \sim \bar{q}_0 + \epsilon\bar{q}_1 + O(\epsilon^2). \tag{7c}$$

Substituting the expansions, Eq. 7, into the system, Eq. 5, and retaining terms of like order, the following lowest order equations arise:

Lowest Order, $O(1)$

$$\frac{d\bar{P}_0}{d\bar{z}} = 0 \tag{8a}$$

$$\bar{P}_0 = \alpha\bar{q}_0^{n_s} \tag{8b}$$

$$\frac{d\bar{Q}_0}{d\bar{z}} = -\bar{q}_0 \tag{8c}$$

$$@\ \bar{z}=0, \quad \bar{Q}_0=1; \quad @\ \bar{z}=1, \quad \bar{Q}_0=0. \tag{8d}$$

For simplicity, before presenting the next-order system, we proceed to solve the lowest order problem, Eq. 8. From Eq. 8a, we find that P_0 is a constant. Thus, from Eq. 8b, we see that the volumetric flow per width, $\bar{q}_0$, is also a constant. Using this fact to integrate Eq. 8c, and applying the boundary conditions, Eq. 8d, the lowest-order solution is given by

$$\bar{q}_0 = 1 \tag{9a}$$

$$\bar{P}_0 = \alpha \tag{9b}$$

$$\bar{Q}_0 = 1 - \bar{z}. \tag{9c}$$

Clearly, in the lowest order, the flow in the die is ideal as given in Eq. 9a. It is the next order, or $O(\epsilon)$, that contains nonuniformity information.

The equations governing the next-order solution are

Next Order, $O(\epsilon)$:

$$\frac{d\bar{P}_1}{d\bar{z}} = -\left[\frac{\bar{Q}_0^{n_c}}{\bar{A}^{(1+3n_c)/2}} + Re\frac{1}{\bar{A}}\frac{d}{d\bar{z}}\left(\frac{\bar{Q}_0^2}{\bar{A}}\right)\right] \tag{10a}$$

$$\bar{P}_1 = \alpha n_s \bar{q}_1 \tag{10b}$$

$$\frac{d\bar{Q}_1}{d\bar{z}} = -\bar{q}_1 \tag{10c}$$

$$@\ \bar{z}=0,1 \quad \bar{Q}_1 = 0. \tag{10d}$$

In writing the system, Eq. 10, we have utilized Eq. 9a explicitly. The system, Eq. 10, is general for any cavity area variation. We note immediately that by integrating Eq. 10c and applying Eq. 10d, we obtain

$$\int_0^1 \bar{q}_1 \, d\bar{z} = 0, \tag{11}$$

indicating that there is no net slot flow in the $O(\epsilon)$ correction.

A particularly convenient and useful cavity area taper equation that is used throughout this article is given in dimensionless form as

$$\bar{A} = (1-\bar{z})^{\gamma}. \tag{12}$$

Such a form, which has been widely used (Cohen and Gutoff, 1992), is chosen to allow the analytic integration of the pressure equation given by Eq. 10a and still retain physically reasonable cavity-area variations. Substituting Eqs. 9c and 12 into Eq. 10a and integrating, the next-order pressure correction in the cavity is thus given by

$$\bar{P}_1 = \frac{1}{\delta}(1-\bar{z})^{\delta} - \frac{Re(2-\gamma)}{\xi}(1-\bar{z})^{\xi} + C, \tag{13a}$$

where

$$\delta = n_c + 1 - \frac{\gamma}{2}(1+3n_c) \tag{13b}$$

$$\xi = 2 - 2\gamma \tag{13c}$$

and C is a constant. Utilizing Eq. 13a in Eqs. 10b and 11, we obtain

$$C = \frac{(2-\gamma)Re}{\xi(\xi+1)} - \frac{1}{\delta(\delta+1)}, \tag{13d}$$

and thus

$$\bar{q}_1 = \frac{1}{\alpha n_s}\left(\frac{1}{\delta}\left[(1-\bar{z})^{\delta} - \frac{1}{\delta+1}\right] - \frac{Re(2-\gamma)}{\xi}\left[(1-\bar{z})^{\xi} - \frac{1}{\xi+1}\right]\right). \tag{13e}$$

The uniformity index N, the measure of flow variations across the die, is directly related to the quantity $\bar{q}_1$; this is seen by substituting Eq. 7c and the lowest order result, Eq. 9a, into Eq. 6 to obtain

$$N \sim \epsilon \bar{q}_1. \tag{13f}$$

The complete slot flow and pressure solutions, valid to $O(\epsilon)$, are thus given by substituting the $O(\epsilon)$ results, Eq. 13, as well as the lowest order results, Eqs. 9a and 9b, into the expansions, Eqs. 7b and 7c.

The error incurred in using the approximate nonuniformity expression of Eq. 13f can be deduced by examining the next-order correction $[O(\epsilon^2)]$ through the use of the expansions, Eq. 7. The solution of the $O(\epsilon^2)$ problem is entirely straightforward (and closely parallels the previously described $O(\epsilon)$ analysis), but the resulting correction is unwieldy. For the case of a Newtonian fluid and a flow in which inertial effects are negligible ($Re = 0$ in Eq. 13), the correction is particularly simple; for this case, the magnitude of the maximum error incurred by using Eq. 13f at any location $\bar{z}$ is $\epsilon^2/45$.

We close this section by noting that for $n_c = n_s = n$, the results (Eq. 13) are precisely those for a power-law fluid flowing through a die. Alternatively, for $n_c = n_s = 1$, results are valid for a purely Newtonian die flow. For cases in which the die flow was purely power law or Newtonian, our analysis is complete at this point. In the next section, we examine the die flow using a combined power-law–Newtonian approach, for which n_c and n_s may have different values.

Switching Criteria: Combination of Power-Law and Newtonian Flows in the Die

In modeling the die flow as a combination of power-law and Newtonian flows, we take advantage of the fact that there are disparate rates of strain in the cavity and slots of the die. In particular, due to the typically small slot height, the rates of strain in the slot will be high; because the cross-sectional characteristic dimension of the cavity is much larger than the slot height, the rates of strain in the cavity will be relatively low. There is the possibility, then, that there are flow conditions where the shear thinning is large enough that the power-law model is totally adequate in the slot, but is physically incorrect in the cavity, where the rates of strain are low enough so that the flow is essentially Newtonian. Our approach is to derive criteria for the cavity and slot, which effectively determine, on average, whether the flow in each of these regions is power law or Newtonian; we will refer to these criteria as *switching criteria*. This allows us to make an appropriate choice for the parameters n_c and n_s in Eq. 13 for the whole cavity and whole slot. Based on the previously described physics, for a given fluid rheology characterized according to Eq. 1 with Eq. 4a, and for a typical die geometry, the following possible physical combinations of Newtonian and power-law flows can arise:

1. The cavity flow is power law. This automatically indicates that the slot flow is power law, based on the geometry of the cavity and slot.
2. The slot flow is Newtonian. This automatically implies that the cavity flow is Newtonian, based on the geometry of the cavity and slot.
3. The cavity flow is Newtonian. For such a case, the slot flow can either be power law or Newtonian.

These three cases suggest that the following switching criteria are necessary:

Cavity Switching Criterion: The slot flow is power law. A criterion is needed to decide if the cavity flow is Newtonian or power law.

Slot Switching Criterion: The cavity flow is Newtonian. A criterion is needed to decide if the slot flow is Newtonian or power law.

We derive these two switching criteria in what follows.

Case A: The cavity switching criterion

The approach taken to determining the cavity switching criterion is to compare the slot nonuniformities arising for the case where the slot flow is power law, and the cavity flow is either Newtonian or power law. By identifying rheological parameters such that the slot-flow nonuniformities for Newtonian and power-law cavity flows are the most similar, a switching criterion can be derived. To begin, we obtain expressions for the slot-flow nonuniformities arising when there is Newtonian or power-law cavity flow (the slot flow is power law in both cases), using Eqs. 13e and 13f. For power-law flow in a slot, $n_s = n$ in Eq. 13e; the parameter α, which is associated with the slot flow, is evaluated accordingly from Eq. 5e. For the case where the cavity flow is power law, $n_c = n$ in Eq. 13, and the parameters Re and ϵ in Eq. 5e are evaluated accordingly. Thus, for the case of power-law slot and cavity flows, the nonuniformity, which we will denote as N_{PL} (to distinguish this result from the Newtonian cavity nonuniformity expression to follow), is given by

$$N_{PL} = \frac{\epsilon}{\alpha n}\left(\frac{1}{\delta}\left[(1-\bar{z})^{\delta} - \frac{1}{\delta+1}\right] - \frac{Re(2-\gamma)}{\xi}\left[(1-\bar{z})^{\xi} - \frac{1}{\xi+1}\right]\right), \quad (14a)$$

where

$$\epsilon = \left(\frac{W^2h^3}{12LA_i^2K^n}\right)\left(\frac{\lambda Q_i}{A_i^{3/2}}\right)^{n-1},$$

$$Re = \left(\frac{\rho\beta Q_i K^n}{\eta_0 W}\right)\left(\frac{\lambda Q_i}{A_i^{3/2}}\right)^{1-n}, \qquad \xi = 2-2\gamma$$

$$\alpha = \frac{2^n}{6}\left(\frac{1}{n}+2\right)^n\left(\frac{\lambda Q_i}{Wh^2}\right)^{n-1},$$

$$\delta = n+1-\frac{\gamma}{2}(1+3n). \quad (14b)$$

Note that in these parameter definitions, the viscous and kinetic shape factors, K and β, are evaluated for the given value of n and the shape of the cavity.

On the other hand, for the case where the cavity flow is Newtonian, $n_c = 1$ in Eq. 13; note that by inspection of Eqs. 13b and 13c, $\delta = \xi$ for a Newtonian cavity. The parameters Re and ϵ, defined in Eq. 5e are also evaluated with $n_c = 1$. We will find it convenient to specifically denote these Newtonian cavity parameters with the subscript N as

$$\epsilon_N = \frac{W^2h^3}{12LA_i^2K_N}, \qquad Re_N = \frac{\rho\beta_N Q_i K_N}{\eta_0 W}, \quad (15a)$$

where K_N and β_N are the viscous and kinetic shape factors, K and β, evaluated for $n=1$, Using this notation, then, the nonuniformity expression for Newtonian flow in the cavity (but power law flow in the slot) is given by

$$N_N = \frac{\epsilon_N}{\alpha n}\left(\frac{1-Re_N(2-\gamma)}{\xi}\left[(1-\bar{z})^{\xi} - \frac{1}{\xi+1}\right]\right), \quad (15b)$$

where ξ and α are given in Eq. 14b.

With these explicit nonuniformity expressions, we now proceed to derive the switching criterion for the cavity. To do so, we first define a distance function, D, which is given by

$$D^2 = \int_0^1 (N_{PL} - N_N)^2\, d\bar{z}. \quad (16)$$

Noting that

$$\epsilon = \left(\frac{K_N}{K^n}\epsilon_N\right)\theta, \qquad Re = \left(\frac{K^n\beta}{K_N\beta_N}Re_N\right)\frac{1}{\theta}, \quad (17a)$$

where

$$\theta = \left(\frac{\lambda Q_i}{A_i^{3/2}}\right)^{n-1}, \quad (17b)$$

the distance function is obtained by substituting Eqs. 14 and 15 into Eq. 16 and integrating

$$\frac{D^2\alpha n}{\epsilon_N^2} = \left(\frac{b^2}{(2\delta+1)(\delta+1)^2}\right)\theta^2 - \left(\frac{2ab}{(\delta+1)(\xi+1)(\xi+\delta+1)}\right)\theta + \left(\frac{a^2}{(2\xi+1)(\xi+1)^2}\right), \quad (17c)$$

where

$$a = 1+Re_N(2-\gamma)\left(\frac{\beta}{\beta_N}-1\right), \qquad b = \frac{K_N}{K^n}. \quad (17d)$$

Then, to determine the switching criterion, we minimize the distance, D, with respect to the parameter θ, holding all other parameters fixed. To do so, we determine the value of θ in Eq. 17c such that

$$\frac{dD^2}{d\theta} = 0. \quad (17e)$$

The result is

$$\theta = \frac{a(2\delta+1)(\delta+1)}{b(\xi+1)(\xi+\delta+1)}. \quad (17f)$$

That this value of θ does minimize D can be seen since D^2, given by Eq. 17c, is a parabola that is concave upward. We note that from the definition of θ in Eq. 17b, a check on the result, Eq. 17f, is that for $n=1$, $\theta = 1$ is obtained.

Substituting definitions of the parameters a, b, and θ in Eq. 17f, and solving for λ, the parameter characterizing the onset of shear thinning (Eq. 4b and Figure 2b), the result is

$$\lambda_c = \frac{A_i^{3/2}}{Q_i}\left(\frac{K_N}{K^n}\right)^{1/(n-1)}\left(1+Re_N(2-\gamma)\left[\frac{\beta}{\beta_N}-1\right]\right)^{1/(n-1)} \times \left(\frac{(2\delta+1)(\delta+1)}{(\xi+1)(\xi+\delta+1)}\right)^{1/(n-1)} \quad (18a)$$

where δ and ξ are given in Eq. 14b. The parameter λ in Eq. 18a has been subscripted with a c to denote that it is the critical value of λ that characterizes the cavity flow. Then for a specified volumetric flow rate, cavity geometry, and rheology characterized by η_0, n, and λ, we make a decision regarding the cavity flow as follows:

$$\text{If } \lambda \le \lambda_c, \text{ the cavity flow is Newtonian.} \tag{18b}$$

$$\text{If } \lambda > \lambda_c, \text{ the cavity flow is power law.} \tag{18c}$$

The result, Eq. 18, is the desired switching criterion in the cavity.

Case B: The slot switching criterion

We now consider the switching criterion for the slot, where the cavity flow is Newtonian, and the slot flow is either Newtonian or power law. We proceed as in the case of the cavity switching criterion (Case A); that is, we obtain a switching criterion by identifying rheological parameters such that the slot-flow nonuniformities for Newtonian and power-law slot flows are the most similar. We begin by writing the nonuniformity expressions for power-law and Newtonian slot flows, with a Newtonian cavity flow. The slot-flow nonuniformity expression for the case of a Newtonian cavity and a power-law slot are already given by Eq. 15b. We will rewrite it below, where we rename this nonuniformity as $N_{PL(\text{slot})}$ for the current calculation:

$$N_{PL(\text{slot})} = \frac{\epsilon_N}{\alpha n}\left(\frac{1 - Re_N(2-\gamma)}{\xi}\left[(1-\bar{z})^{\xi} - \frac{1}{\xi+1}\right]\right), \tag{19}$$

where the parameters ξ and α are given in Eq. 14b, and Re_N and ϵ_N are defined in Eq. 15a. The nonuniformity expression for the case of Newtonian slot and cavity flows, denoted as $N_{N(\text{slot})}$, can be obtained by choosing $n_c = n_s = 1$ in Eq. 13 and the parameter definitions, Eq. 5e. The result is

$$N_{N(\text{slot})} = \epsilon_N\left(\frac{1 - Re_N(2-\gamma)}{\xi}\left[(1-\bar{z})^{\xi} - \frac{1}{\xi+1}\right]\right). \tag{20}$$

As in the case of the cavity switching criterion, we minimize the distance between Eqs. 19 and 20. By inspection, it is apparent that the distance between $N_{PL(\text{slot})}$ and $N_{N(\text{slot})}$ can be made precisely zero for all widthwise locations $\bar{z}$ by choosing:

$$\alpha n = 1. \tag{21}$$

Using the definition of α from Eq. 5e (with $n_s = n$) and solving for λ, the result is

$$\lambda_s = \frac{Wh^2}{Q_i}\left[\frac{6}{n2^n\left(\dfrac{1}{n}+2\right)^n}\right]^{1/(n-1)}, \tag{22a}$$

where we have subscripted the value of λ with an s to denote that it is critical value in the slot. The slot switching criterion is thus applied as follows. For a specified value of n, volumetric flow rate, die width, and slot height, we compare the value of λ for the given rheology to λ_s:

$$\text{If } \lambda \le \lambda_s, \text{ the slot flow is Newtonian.} \tag{22b}$$

$$\text{If } \lambda > \lambda_s, \text{ the slot flow is power law.} \tag{22c}$$

The result, Eq. 22, is the desired switching criterion in the slot.

Before leaving our derivation of the slot switching criterion, we examine the impact of the criterion on the cavity pressure evaluation itself. In particular, the pressure expansion, Eq. 7b, coupled with Eq. 9b, indicates that to lowest order

$$\bar{P} \sim \alpha, \tag{23}$$

where α is defined in Eq. 5e. Recall that $\bar{P}$ is made dimensionless with the Newtonian pressure drop across the slot for the case of perfect slot-flow uniformity evaluated at the zero shear viscosity η_0. For a Newtonian fluid, $n_s = 1$, and thus $\alpha = 1$ from Eq. 5e. When there is shear thinning in the slot, $n_s = n$ and $n < 1$, and clearly the parameter α must be less than 1: the loss in viscosity due to shear thinning reduces the pressure drop across the slot. Thus, a physical bound on the parameter α for a shear thinning fluid, with regards to pressure, is $\alpha \le 1$. At this point, we note that the slot switching criterion is derived based on a rearrangement of Eq. 21, which assures that, when the switch from Newtonian to power-law behavior occurs, the predicted slot-flow nonuniformities are identical for both cases. Equation 21 implies that $\alpha = 1/n_s$; when Eq. 22b is satisfied, $n_s = 1$; when Eq. 22c is satisfied, $n_s = n$. At $\lambda = \lambda_s$, then, the lowest order pressure, given by Eq. 23, jumps abruptly from $\bar{P} \sim 1/n$ to $\bar{P} \sim 1$. We note also that for $n < 1$, the power-law pressure solution $\bar{P} \sim 1/n$ is not physically possible, in that for $n < 1$, the cavity pressure is greater than that for the Newtonian case.

To remedy this situation, another possible choice of criterion to switch from Newtonian to power-law behavior in the slot might be

$$\alpha = 1. \tag{24a}$$

This choice guarantees that the lowest order cavity *pressure* would be continuous in switching from Newtonian to power-law behavior in the slot. The critical value of λ corresponding to Eq. 24a is

$$\lambda_{s_p} = \frac{Wh^2}{Q_i}\left[\frac{6}{2^n\left(\dfrac{1}{n}+2\right)^n}\right]^{1/(n-1)}, \tag{24b}$$

where we have subscripted λ_s with a "p" to denote that this value assures a continuity in pressure. However, since Eq. 24a is satisfied, Eq. 21 is not; consequently, although the pressure solution is now continuous in switching from power-law to Newtonian slot-flow models, the nonuniformity result is discontinuous in this transition. Thus, we see that one limitation of the switching approach taken for the slot is that it is not possible to switch from Newtonian to power-law behavior and obtain a continuous transition in both the nonuniformity and pressure solutions. In this article, we choose to focus our analysis on predictions of flow nonuniformity and *not* pressure; for this reason, the criterion, Eq. 22, is utilized. The validity of making this choice is borne out via comparisons with the more exact Carreau model predictions (which do not require a switching approach) presented later in this article. It is safe to say, however, that for $\lambda_s < \lambda < \lambda_{s_p}$, there is some ambiguity in modeling the slot flow as being purely Newtonian or purely power law; it is likely that in this range of λ the smooth transition between the purely power-law and Newtonian regions in Figure 2b is occurring, and neither model is entirely adequate.

Summary: Prediction of die performance using switching criteria and asymptotic nonuniformity expression

Before leaving this section, we now summarize how the switching criteria are used, in conjunction with the previously derived nonuniformity expressions via asymptotics, to predict flow nonuniformities in an extrusion die. The equation governing flow nonuniformities, N, is given by Eqs. 13e and 13f, where parameter definitions are given in Eqs. 13b, 13c, and 5e; these are defined in terms of the parameters n_c and n_s, which are chosen according to the switching criteria, Eqs. 18 and 22. It is possible to combine these two switching criteria into a more concise form by noting that not all combinations of Newtonian and power-law slot and cavity flows are physically possible; the possible physical combinations have already been given in statements 1–3 in the introductory discussion directly preceding our derivation of the switching criteria. Thus, for a given die geometry and fluid rheology characterized by Eq. 1 with Eq. 4a (requiring the specification of λ, n, and η_0), the switching criteria are combined to yield:

If $\lambda \geq \lambda_c$, cavity and slot flows are power law: $n_c = n_s = n$. (25a)

If $\lambda \leq \lambda_s$, cavity and slot flows are Newtonian: $n_c = n_s = 1$. (25b)

If $\lambda_s < \lambda < \lambda_c$, cavity flow is Newtonian; slot flow is power law: $n_c = 1$, $n_s = n$. (25c)

These are the mathematical equivalent to statements 1–3 referred to earlier. In Eq. 25, λ_c and λ_s are given by Eqs. 18a and 22a, respectively. For typical die geometries, λ_c is greater than λ_s, which is consistent with the fact that higher rates of strain are found in the slot than in the cavity. Note that in the parameter definitions given in Eq. 5e, the values of the shape factors K and β are dependent on the effective fluid rheology in the cavity and are evaluated based on the value of n_c obtained via Eq. 25.

Assessment of the Combined Power-Law and Newtonian Approach

We now proceed to assess the validity of our Newtonian–power-law analysis with switching criteria (we will refer to this approach as the *combined approach* in what follows). To do so, we will compare results of our combined approach with those obtained using the Carreau model, Eq. 2, implemented using one-dimensional modeling assumptions. The Carreau model is a natural choice for our comparison, as we have specifically related the Carreau parameters to the power-law model used in our combined approach according to Eq. 4a. As discussed in the introduction, the Carreau model allows for a smooth linkage between the constant viscosity Newtonian regions and the linear power-law region of the rheological curve (Figure 2), and often fits experimental rheological curves quite well. Thus, when the Carreau model is used in theoretical predictions, there is no need to consider the switching procedures derived previously. When implemented using the same one-dimensional assumptions used to obtain the system (Eq. 5), solutions obtained using the Carreau model can be viewed as "exact" for comparison with our combined approach. As demonstrated by Weinstein and Ruschak (1996), the Carreau rheology cannot be implemented *easily* in a one-dimensional model since constant cavity shape factors cannot be determined (this is what leads us to the combined Newtonian–power-law approach in the first place); nevertheless, it can be done. In what follows, we first discuss the one-dimensional implementation of the Carreau model. Subsequent to this, we compare Carreau and combined model results.

Because of the complications associated with the implementation of the Carreau model, which will become evident in the following discussion, we restrict attention to a geometrically simple extrusion die, consisting of a cavity of constant circular cross-sectional area and constant slot dimensions. Additionally, we will neglect fluid inertia associated with the cavity flow. With these assumptions, we discuss how the system, Eq. 5, valid for power-law or Newtonian fluids, is altered for a Carreau model fluid. In the current problem in which fluid inertia is neglected in the cavity, the 1-D approach incorporates the assumption that the cavity flow rate, and hence pressure gradient, slowly varies in the z-direction (Figure 1a) as fluid "leaks" into the slot. The flow leakage effect is incorporated via the mass balance of Eq. 5c, and this balance is the same regardless of the rheological model chosen. Despite the depletion of flow from the cavity, the pressure-drop–flow-rate relationship in the cavity is assumed to have a fully developed form; for Newtonian and power-law fluids in a circular cavity, the pressure-drop flow relationship is obtainable in closed form as given by Eq. 5a (with the appropriate viscous shape factor K, and $Re = 0$ based on the assumption of negligible fluid inertia). For a Carreau model fluid, the flow relationship equivalent to Eq. 5a cannot be obtained in closed form and must be numerically determined. Additionally, in the 1-D approach, the pressure drop and flow relationship in the slot is assumed to be that obtained for unidirectional flow between parallel walls, despite the fact that the slot flow can vary due to pressure variations in the widthwise z-direction (Figure 1a). As for the case of the cavity flow, the pressure-drop and flow relationship is analytical for Newtonian and power-law fluids and is given by Eq. 5b; the Carreau model relationship in the slot must be determined numerically as well. The boundary conditions, Eq. 5d, are applicable regardless of the rheological model.

To solve the just-described Carreau system, the flow in the slot is considered first. The velocity field is discretized across the slot height (i.e., h in Figure 1b), and a finite difference approximation is used for all derivatives. The volumetric flow rate per width in the slot is imposed, and provides an integral constraint involving the velocity that is discretized using the trapezoidal rule. Newton's method is employed to solve the resulting set of nonlinear algebraic equations, and the pressure drop across the slot is determined. This procedure is repeated for a range of imposed volumetric flow rates per width; a fourth-order polynomial is then fitted to these results, expressing the volumetric flow rate per width as a function of the pressure drop across the slot (which determines the cavity pressure). Note that this polynomial relationship must be recalculated as flow conditions, slot height, and fluid rheology are varied (in contrast to the corresponding analytical relationship, Eq. 5b, for a power-law fluid). In the cavity, the fully developed equation relating the velocity and pressure is discretized using a finite difference approximation for all derivatives; the velocity field is discretized both radially and axially (i.e., in the z-direction in Figure 1a), and the cavity pressure is discretized axially. An additional relationship between the cavity pressure and velocity field is obtained by substituting the fitted polynomial described earlier for the slot flow into the mass balance, Eq. 5c, and expressing the volumetric flow rate as the integral of the velocity field; this expression is subsequently discretized as for the fully developed equation, employing the trapezoidal rule for the integral and a finite difference approximation for the derivatives. The resulting discretized equations, coupled with the boundary conditions discussed in the previous paragraph, provide a well-posed nonlinear algebraic set of equations. These equations are solved via Newton's method to determine the variation in slot flow with axial location z.

We now compare Carreau results with those of the one-dimensional approach using the combined Newtonian power-law models and switching criteria. For the purposes of demonstrating the validity of our approach, we will find it useful to work in dimensional terms. The die geometrical and flow parameters common to all presented results are

$$A_i = 2.7\ \text{cm}^2, \quad L = 2\ \text{cm}, \quad h = 0.0254\ \text{cm}, \quad W = 66\ \text{cm}, \quad Q_i = 33\ \text{cm}^3/\text{s}. \tag{26a}$$

Since the cavity area is constant, Eq. 12 indicates that

$$\gamma = 0, \tag{26b}$$

and since fluid inertial effects are neglected in the cavity,

$$Re = 0 \text{ in Eq. 13} \quad \text{and} \quad Re_N = 0 \text{ in the cavity switching criterion, Eq. 18.} \tag{26c}$$

We will consider results for $n = 0.5$ and $n = 0.8$ in what follows; viscous shape factors, K, for the circular cavity shape and these power-law indices are (from Lee and Liu, 1989):

$$K = 0.027243 \text{ for } n = 0.8; \quad K = 0.00898 \text{ for } n = 0.5. \tag{26d}$$

For the purposes of comparing our analysis with the results of the Carreau model, the assumption associated with Eq. 4a, $\eta \gg {}_{h\infty}$, has been implicitly used in our development of the combined model. Thus, we choose ${}_{h\infty} = 0$ in the Carreau model (Eq. 2) for all presented results. Additionally, since we are neglecting fluid inertial effects in the cavity, all nonuniformity results presented are independent of the zero shear viscosity η_0 for both the combined and Carreau models (see Eq. 13 with $Re = 0$ for the combined model).

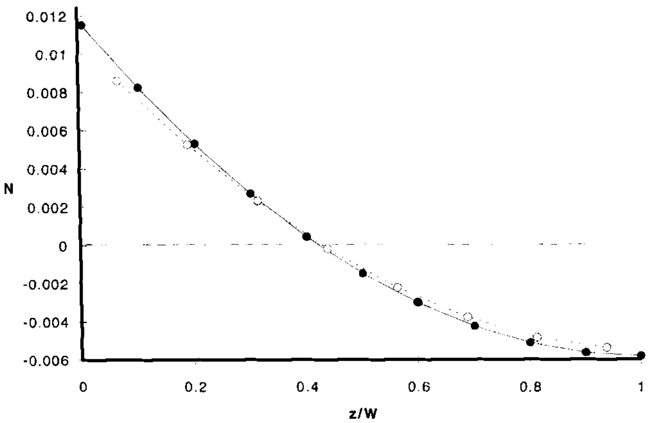

Figure 3. Theoretical predictions of widthwise nonuniformity, N, from the combined power-law–Newtonian model and the Carreau model for $\lambda = \lambda_c$.

The single-cavity die has a constant slot length and constant circular cross-sectional area, and there is negligible fluid inertia. Geometrical and flow parameters are given in Eq. 26, with $n = 0.8$ and $\lambda = 0.0422$ s. Combined model —●—; Carreau model ---○---.

Figure 3 gives predictions for the widthwise nonuniformity, N (as defined in Eq. 13), from the combined power-law–Newtonian model with switching (labeled as "Combined" on the figure) and the numerically implemented Carreau model, for $\lambda = 0.0422$ s, and $n = 0.8$. We note that this value of λ is precisely the switching value in the cavity, that is, $\lambda = \lambda_c$ in our combined model evaluated according to Eq. 18a; the flow in the slot is power law for this value according to Eq. 25. It is seen that the nonuniformity predictions of the Carreau model and the combined approach agree fairly well, with the combined model slightly overpredicting the nonuniformities. Figure 4 gives nonuniformity results for $n = 0.8$, except that now the values of λ have been varied by a factor of 10 from $\lambda = \lambda_c$. Clearly, for $\lambda \ll \lambda_c$ and $\lambda \gg \lambda_c$, the Carreau and combined model predictions are virtually indistinguishable (the slot is still power law for these cases). Figures 5 and 6 give similar comparisons to those in Figures 3 and 4, where now $n = 0.5$; for all cases shown, the slot flow is power law. Clearly, for $\lambda = \lambda_c$ in Figure 5, the combined model provides a larger overprediction of the Carreau nonuniformities than for the $n = 0.8$ comparison in Figure 3. Note, however, for $\lambda \ll \lambda_c$ and $\lambda \gg \lambda_c$ in Figure 6, the Carreau and combined model results agree almost exactly (Figure 4).

Figures 7 and 8 give similar comparisons to those in the previous figures, except now the cavity flow is Newtonian, and attention is focused on predictions near the switching value of λ in the slot. Figure 7 gives results for $n = 0.8$ and $\lambda = \lambda_s$. Note that the nonuniformities predicted from the combined model and Carreau model are in good agreement. Figure 8 gives results for $n = 0.8$, except that now the values of λ have been varied by a factor of 10 from $\lambda = \lambda_s$. The agreement

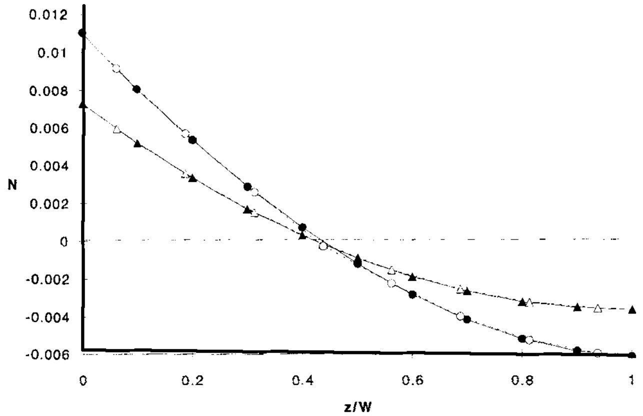

Figure 4. Theoretical predictions of widthwise nonuniformity, N, from the combined power-law–Newtonian model and the Carreau model for $\lambda < \lambda_c$ and $\lambda > \lambda_c$.

All conditions are identical to those in Figure 3, except that the values of λ are different. Results for $\lambda = 0.00422$ s $< \lambda_c$: combined model —▲—, Carreau model ---△---. Results for $\lambda = 0.422$ s $> \lambda_c$: combined model —●—, Carreau model ---○---.

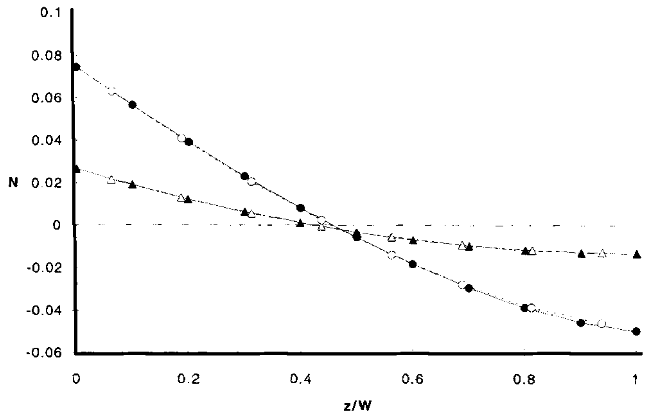

Figure 6. Theoretical predictions of widthwise nonuniformity, N, from the combined power-law–Newtonian model and the Carreau model for $\lambda < \lambda_c$ and $\lambda > \lambda_c$.

All conditions are identical to those in Figure 5, except that the values of λ are different. Results for $\lambda = 0.00432$ s $< \lambda_c$: combined model —▲—; Carreau model ---△---. Results for $\lambda = 0.432$ s $> \lambda_c$: combined model —●—; Carreau model ---○---.

between the Carreau model and combined models is even better than in Figure 7. Figures 9 and 10 give analogous comparisons to those in Figures 7 and 8, except now $n = 0.5$. For all cases, agreement between the predictions of the Carreau and combined models is good.

We now qualitatively explain the trends in these figures. To begin, we focus on the nonuniformity results of Figures 3 and 4. We note that for values of λ significantly greater or less than λ_c, the assumption that the cavity is power law or Newtonian, respectively, is quite good. Using this intuition, it is anticipated that compared with the more exact Carreau model, the greatest error in using the combined approach of switching from Newtonian to power-law behavior in the cavity is incurred for values of λ equal to λ_c. This is analogous to the comparison of the truncated power-law model with the experimental curve shown in Figure 2b, which we can view as being well represented by the Carreau model. In the figure, the largest distance between the viscosity predictions of the Carreau and truncated models arises at the critical rate of strain $|\dot{\gamma}|_c$, which is related to λ via Eq. 4b as $\lambda = 1/|\dot{\gamma}|_c$; this value of λ can be viewed as being directly analogous to λ_c in our combined approach. The inaccuracies in viscosity predictions afforded by the truncated model translate into corresponding inaccuracies in pressure and flow predictions in the extrusion die. The important point to note here is that when the Carreau and combined models do not precisely agree, the combined approach appears to overpredict the flow nonuniformities, which is a safe position from an engineering perspective. Analogous arguments can be made for the results of Figures 5 and 6, where the validity of the switching criterion

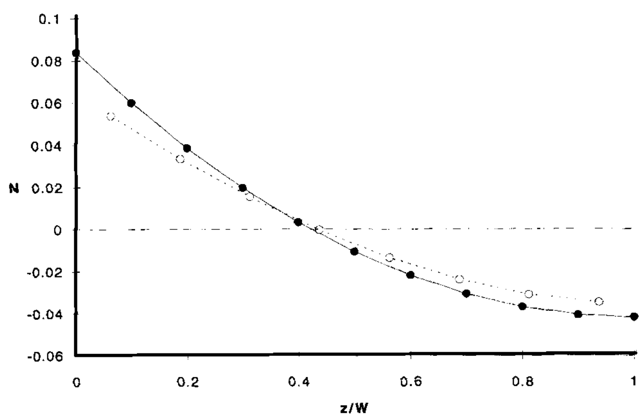

Figure 5. Theoretical predictions of widthwise nonuniformity, N, from the combined power-law–Newtonian model and the Carreau model for $\lambda = \lambda_c$.

The single-cavity die has a constant slot length and constant circular cross-sectional area, and there is negligible fluid inertia. Geometrical parameters are given in Eq. 26, with $n = 0.5$ and $\lambda = 0.0432$ s. Combined model —●—; Carreau model ---○---.

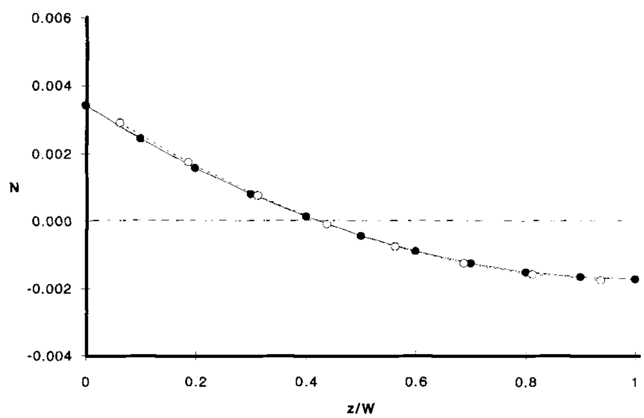

Figure 7. Theoretical predictions of widthwise nonuniformity, N, from the combined power-law–Newtonian model and the Carreau model for $\lambda = \lambda_s$.

The single-cavity die has a constant slot length and constant circular cross-sectional area, and there is negligible fluid inertia. Geometrical parameters are given in Eq. 26, with $n = 0.8$ and $\lambda = 9.71 \times 10^{-5}$ s. Combined model —●—; Carreau model ---○---.

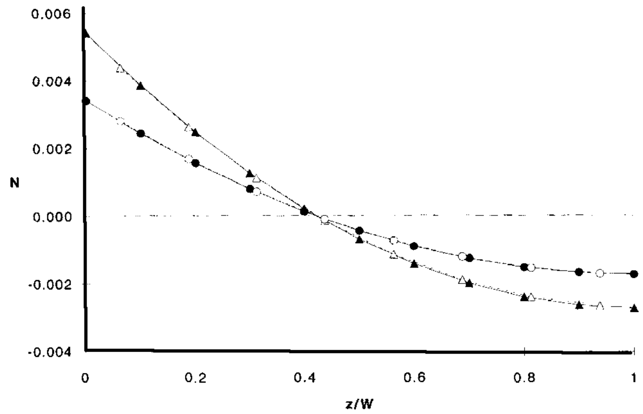

Figure 8. Theoretical predictions of widthwise nonuniformity, N, from the combined power-law–Newtonian model and the Carreau model for $\lambda < \lambda_s$ and $\lambda > \lambda_s$.

All conditions are identical to those in Figure 7, except that the values of λ are different. Results for $\lambda = 9.71 \times 10^{-6}$ s $< \lambda_s$: combined model —▲—; Carreau model ---△---. Results for $\lambda = 9.71 \times 10^{-4}$ s $> \lambda_s$: combined model —●—; Carreau model ---○---.

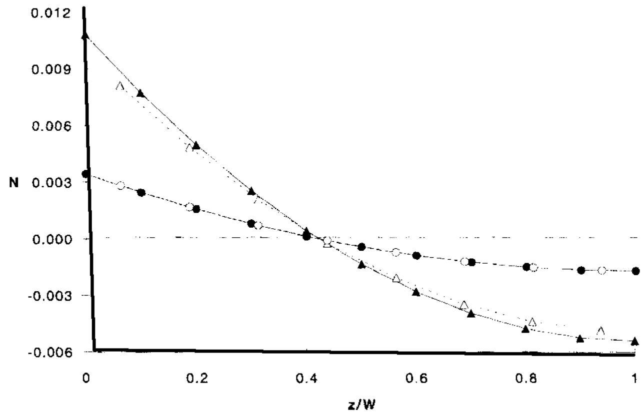

Figure 10. Theoretical predictions of widthwise nonuniformity, N, from the combined power-law–Newtonian model and the Carreau model for $\lambda < \lambda_s$ and $\lambda > \lambda_s$.

All conditions are identical to those in Figure 9, except that the values of λ are different. Results for $\lambda = 7.17 \times 10^{-6}$ s $< \lambda_s$: combined model —▲—; Carreau model ---△---. Results for $\lambda = 7.17 \times 10^{-4}$ s $> \lambda_s$: combined model —●—; Carreau model ---○---

in the slot is examined. Note that the data comparisons in Figure 10 do not quite follow the trend that the agreement between the combined and Carreau model predictions is better for $\lambda \gg \lambda_s$ than it is at $\lambda = \lambda_s$. It is possible that this behavior may be due to small numerical errors in the Carreau predictions; we were unable to refine the Carreau numerical predictions further due to the size of the numerical system to be solved.

Before closing this section, we make an important comment regarding the switching criteria themselves, which impacts the preceding comparisons of results of the Carreau model and combined approach just given. We note that at $\lambda = \lambda_c$, there is an indeterminate condition where the flow in the cavity is *either* Newtonian or power law, while the slot flow is power law. The critical value λ_c for the cavity flow is derived such that it minimizes the "distance" between the nonuniformities in both cases widthwise across the die; in general, this distance is nonzero. Thus, for flows that have $\lambda = \lambda_c$, there are really two possible nonuniformity predictions from the combined power-law Newtonian approach. Figures 11 and 12 give nonuniformity predictions using the combined approach for $\lambda = \lambda_c$, for the same conditions as in Figures 3 and 5, respectively. In each figure, the two possible nonuniformity predictions are given (one for a Newtonian cavity flow, and the other for a power-law cavity flow; both are with a power-law slot flow). Comparing Figures 11 and 12, it is seen that as n increases, the deviations between the two curves increases; nevertheless, it is clear that on average, the distance between the two curves pointwise across the domain is very small, which is consistent with our derivation of λ_c. In our comparisons with the Carreau model predictions in Figures 3 and 5, note that we have utilized the Newtonian

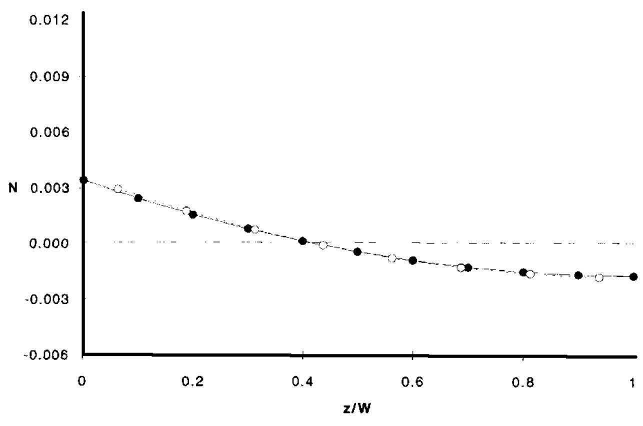

Figure 9. Theoretical predictions of widthwise nonuniformity, N, from the combined power-law–Newtonian model and the Carreau model for $\lambda = \lambda_s$.

The single-cavity die has a constant slot length and constant circular cross-sectional area, and there is negligible fluid inertia. Geometrical parameters are given in Eq. 26, with $n = 0.5$ and $\lambda = 7.17 \times 10^{-5}$ s. Combined model —●—; Carreau model ---○---.

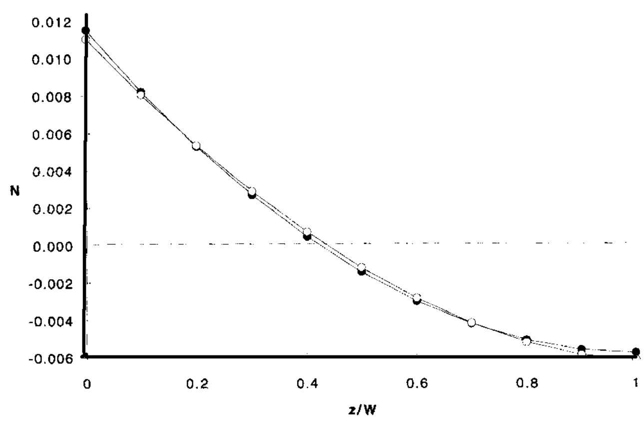

Figure 11. Two theoretical predictions of widthwise nonuniformity, N, from the combined power-law–Newtonian model $\lambda = \lambda_c$.

All conditions are identical to those in Figure 3. Results for Newtonian cavity flow —●—; results for power-law cavity flow —○—.

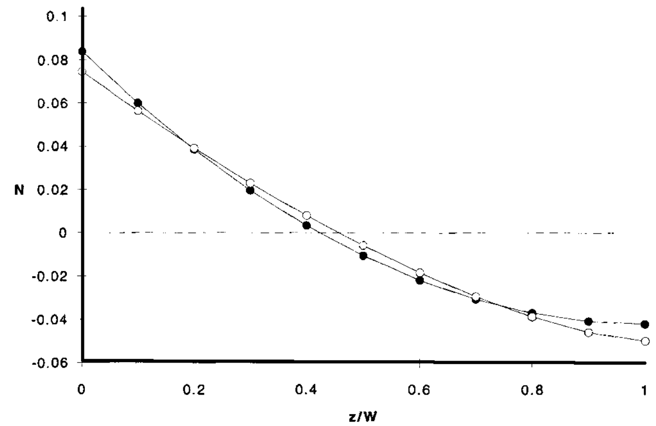

Figure 12. Two theoretical predictions of widthwise nonuniformity, *N*, from the combined power-law–Newtonian model $\lambda = \lambda_c$.

All conditions are identical to those in Figure 5. Results for Newtonian cavity flow —●—; results for power-law cavity flow —○—.

cavity results of the combined model given in Figures 11 and 12, respectively. It is understood, then, that there is some uncertainty in the pointwise profiles predicted by the combined model at λ_c. On the other hand, when a switch from power-law to Newtonian flow occurs in the slot at $\lambda = \lambda_s$ (the cavity flow is Newtonian for both cases), there is no such uncertainty in the nonuniformity pointwise profiles (in the context of the combined approach). This is because the switching criterion, as derived, always yields a situation where the distance between the power-law and Newtonian slot-flow cases is identically zero for all widthwise locations in the die.

Summary and Conclusions

The flow through a single-cavity extrusion die has been considered through the use of one-dimensional equations. We have demonstrated how linearization can lead to significant simplifications in the analysis of die flow, and how this linearization can be accomplished using formal asymptotics. In contrast to previous analyses, which have generally been numerical, we have obtained an analytic solution to predict widthwise flow nonuniformities in a die having a specific cavity area variation and in which cavity inertial effects are important; this solution can be applied for both power-law and Newtonian fluids. We have proposed a novel approach to modeling the flow of shear-thinning fluids in a die, involving the combination of Newtonian and power-law flows in the cavity and slot. The validity of this approach has been demonstrated through comparison with numerical results of a one-dimensional die model for a Carreau fluid.

As discussed by Weinstein and Ruschak (1996), constant shape factors cannot be identified for a Carreau model fluid; it was stated that this makes it extremely difficult to utilize the Carreau model in the context of a 1-D approach. Even for the simple geometry examined in this article, the Carreau model analysis is numerically intensive. This further demonstrates the need for a truncated power-law approach, such as given by Yuan (1995), or a combined approach such as given in this article, to model shear-thinning fluids in a die when using a one-dimensional approach. In this way, constant shape factors can be identified (as a function of cavity shape and power-law index), and one-dimensional equations, as given in Eq. 5, can be utilized.

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